Chapter 3
Risk-Neutral Agent

When a risk-neutral agent accepts a contract offer (w, p), his expected utility rate is composed of the expected value of the compensation rate from the principal and a deterministic cost rate of the service capacity which can be expressed as $w-pP(1)-\mu$, where $P(1)$ denotes the steady state probability of the unit being in the failed state. Similarly denote the steady state probability of the unit being operational by $P(0) = 1 - P(1)$.

Notation: $(x)_+ = x$ when $x \geq 0$ and $(x)_+ = 0$ when $x < 0$.

A risk-neutral agent's expected utility rate is:

$$u_A(\mu; w, p) = (w - pP(1) - \mu)_+ \text{ for } w > 0, p > 0, \mu \geq 0 \tag{3.1}$$

$P(0)$ and $P(1)$ (functions of λ and μ), represent the proportion of time in the steady state the Markov process is in state 0 and state 1 respectively (Ross 2006). They satisfy the balance equations of the Markov process and sum up to 1, thus $P(0) = \mu/(\lambda + \mu),\ P(1) = \lambda/(\lambda + \mu)$:

$$u_A(\mu; w, p) = \left(w - \frac{p\lambda}{\lambda + \mu} - \mu\right)_+ \text{ for } w > 0, p > 0, \mu \geq 0 \tag{3.2}$$

Since the principal determines w and p she can always entice the agent to accept the contract.

For $r > 0$ (determined exogenously by the market), the principal's expected profit rate is composed of the expected revenue rate generated by her unit, the expected penalty rate collected from the agent and the compensation rate paid to the agent:

S. Zeng, M. Dror, *Formulating Principal-Agent Service Contracts for a Revenue Generating Unit*, SpringerBriefs in Operations Management,
DOI 10.1007/978-3-319-18672-6_3

$$\Pi_P(w,p;\mu) = rP(0) - w + pP(1) = \frac{r\mu}{\lambda+\mu} - w + \frac{p\lambda}{\lambda+\mu}$$
$$\text{for } w > 0, p > 0, \mu \geq 0 \tag{3.3}$$

Observation 3.1. *We note that under another type of contract, where the principal compensates the agent only for each unit of uptime (instead of each unit of time), the agent's expected utility rate is equivalent to (3.2), and the principal's expected profit rate is equivalent to (3.3): Under the new type of contract, denote the compensation rate by $\tilde{w}$ and the penalty rate by $\tilde{p}$, therefore the agent's expected utility rate becomes:*

$$u_A(\mu;\tilde{w},\tilde{p}) = (\tilde{w}P(0) - \tilde{p}P(1) - \mu)_+ = \left(\frac{\tilde{w}\mu}{\lambda+\mu} - \frac{\tilde{p}\lambda}{\lambda+\mu} - \mu\right)_+$$
$$\text{for } \tilde{w} > 0, \tilde{p} > 0, \mu \geq 0 \tag{3.4}$$

and the principal's expected profit rate becomes:

$$\Pi_P(\tilde{w},\tilde{p};\mu) = rP(0) - \tilde{w}P(0) + \tilde{p}P(1) = \frac{r\mu}{\lambda+\mu} - \frac{\tilde{w}\mu}{\lambda+\mu} + \frac{\tilde{p}\lambda}{\lambda+\mu}$$
$$\text{for } \tilde{w} > 0, \tilde{p} > 0, \mu \geq 0 \tag{3.5}$$

Replacing $\tilde{w}$ by w and $\tilde{p}$ by $(p - w)$ in (3.4) and (3.5) we obtain (3.2) and (3.3) respectively.

Note that a performance based contract can even take the form such that a compensation rate is specified for each unit of uptime (instead of each unit of time) and no penalty rate is charged whatsoever. That is, the principal controls only one variable (the compensation rate) instead of two (the compensation rate and the penalty rate). However this form of performance based contract is not discussed in this work.

Returning to the agent as in (3.2) we define the part inside the brackets by

$$u(\mu) \equiv w - \frac{p\lambda}{\lambda+\mu} - \mu \tag{3.6}$$

i.e., for $\mu \geq 0$, $u(\mu)$ is continuous and differentiable everywhere:

$$\frac{du(\mu)}{d\mu} = \frac{p\lambda}{(\lambda+\mu)^2} - 1 \text{ and } \frac{d^2u(\mu)}{d\mu^2} = -\frac{2p\lambda}{(\lambda+\mu)^3} < 0$$
$$u(0) = w - p,\ \left.\frac{du(\mu)}{d\mu}\right|_{\mu=0} = \frac{p}{\lambda} - 1 \text{ and } \lim_{\mu\to+\infty} \frac{du(\mu)}{d\mu} = -1$$

3.1 Optimal Strategies for Risk-Neutral Agent

Note that $u(\mu)$ in (3.6) increases and $\Pi_P(w,p;\mu)$ in (3.3) decreases in w, therefore for any value of penalty rate p, the principal can raise her expected profit rate by adjusting the rate w low enough while ensuring the agent's participation by setting the agent's expected utility rate equal to his reservation utility rate. Although the principal cannot contract directly on the agent's capacity, she presumes the agent will optimize his expected utility rate. That is, for any compensation rate w and penalty rate p proposed by the principal, the agent computes the value of μ that maximizes his expected utility rate and decides whether to accept the contract or not by solving the following optimization problem:

$$\max_{\mu\geq 0}\ u(\mu) = \max_{\mu\geq 0}\left\{w - \frac{p\lambda}{\lambda+\mu} - \mu\right\} \tag{3.7}$$

with agent's optimal service capacity denoted by $\mu^*(w,p) = \text{argmax}_{\mu\geq 0}\, u(\mu)$.

We describe the agent's optimal response to any possible contract offer $(w,p) \in \mathbb{R}^2_+$ in Proposition 3.3, but we start with a simple technical lemma – one of many.

Lemma 3.2. *If $p > \lambda > 0$, then $p > 2\sqrt{p\lambda} - \lambda > 0$.*

Proof. If $p > \lambda > 0$, then $2\sqrt{p\lambda} - \lambda > 2\lambda - \lambda = \lambda > 0$ and $p - 2\sqrt{p\lambda} + \lambda = \left(\sqrt{p} - \sqrt{\lambda}\right)^2 > 0$, where the latter inequality indicates $p > 2\sqrt{p\lambda} - \lambda$. □

Proposition 3.3. *Consider a risk-neutral agent with $u_A(\mu; w,p)$ given in (3.2).*

(a) *Given $p \in (0,\lambda]$, then the agent accepts the contract only when $w \geq p$ and does not commit any service capacity ($\mu^*(w,p) = 0$) resulting in expected utility rate $u_A(\mu^*(w,p); w,p) = w - p \geq 0$.*
(b) *Given $p > \lambda$, then the agent accepts the contract only when $w \geq 2\sqrt{p\lambda} - \lambda$ and installs service capacity $\mu^*(w,p) = \sqrt{p\lambda} - \lambda > 0$ resulting in expected utility rate $u_A(\mu^*(w,p); w,p) = w - 2\sqrt{p\lambda} + \lambda \geq 0$.*

Proof. Figure 3.1 illustrates the form of $u(\mu)$ when the value of p falls in different ranges. The structure of the proof for Proposition 3.3 is depicted in Fig. 3.2.

Case $p \in (0,\lambda]$: $u(\mu)$ is decreasing for $\mu \geq 0$, therefore the optimal service capacity is set at $\mu^*(w,p) = 0$ and $u(\mu^*(w,p)) = w - p$.

Subcase $w \in (0,p)$: $u(\mu^*(w,p)) < 0$, therefore the agent rejects the contract.
Subcase $w \geq p$: $u(\mu^*(w,p)) \geq 0$, thus the agent would accept the contract if offered.

Case $p > \lambda$: The service capacity that maximizes $u(\mu)$ is positive as seen from the first order condition $du(\mu)/d\mu|_{\mu=\mu^*(w,p)} = 0 \Rightarrow \mu^*(w,p) = \sqrt{p\lambda} - \lambda > 0$ and $u(\mu^*(w,p)) = w - 2\sqrt{p\lambda} + \lambda$. According to Lemma 3.2 we have to resolve the following subcases:

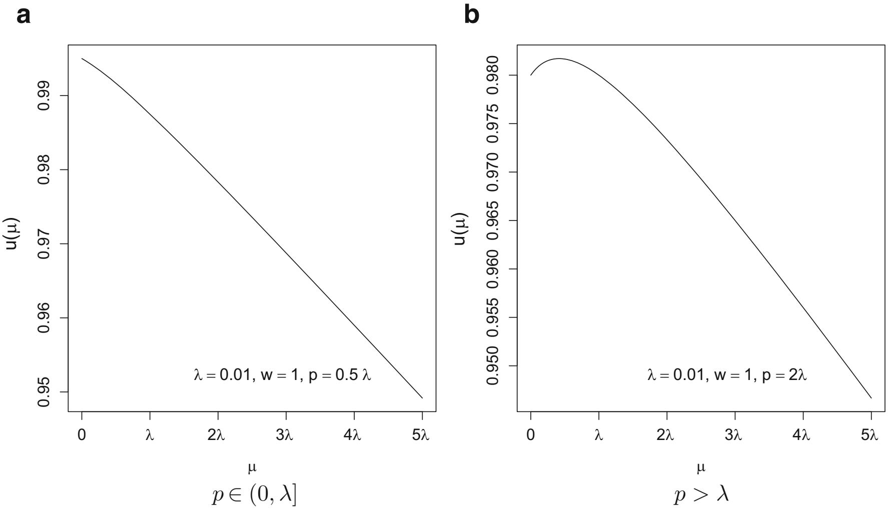

Fig. 3.1 Illustration of the forms of $u(\mu)$

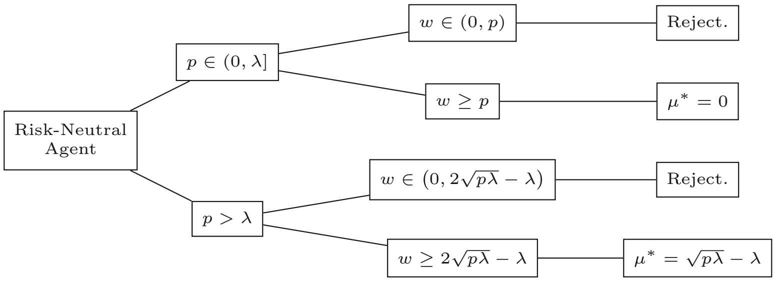

Fig. 3.2 Structure of the proof for Proposition 3.3

Subcase $w \in \left(0, 2\sqrt{p\lambda} - \lambda\right)$: $u(\mu^*(w, p)) < 0$, therefore the agent rejects the contract.

Subcase $w \geq 2\sqrt{p\lambda} - \lambda$: $u(\mu^*(w, p)) \geq 0$, therefore the agent would accept the contract if offered.

□

In summary, given exogenous market conditions such that there exists a contract benefiting both the agent and principal (see Theorem 3.4 later), only one formula is necessary for the agent to determine his service capacity: $\mu^*(w, p) = \sqrt{p\lambda} - \lambda > 0$.

The conditions when the agent accepts the contract are depicted by the shaded areas in Fig. 3.3. The two shaded areas of different grey scales represent conditions $\{(w, p) : p \in (0, \lambda], w \geq p\}$ and $\left\{(w, p) : p > \lambda, w \geq 2\sqrt{p\lambda} - \lambda\right\}$ under which the agent accepts the contract but responds differently. The lower bound function of the

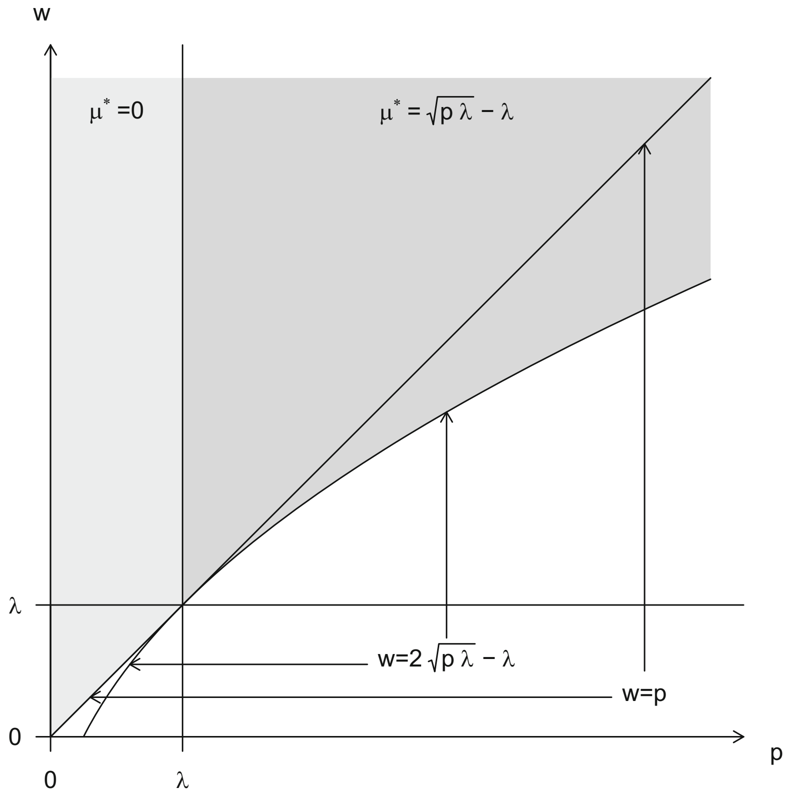

Fig. 3.3 Conditions when a risk-neutral agent accepts the contract

shaded areas (denoted by $w_0(p)$) represents the contract offers that result in agent zero expected utility rate. $w_0(p)$ is defined as follows:

$$w_0(p) = \begin{cases} p \text{ for } p \in (0, \lambda] \\ 2\sqrt{p\lambda} - \lambda \text{ for } p > \lambda \end{cases}$$

Note that since $\lim_{p\to\lambda^-} w_0(p) = \lim_{p\to\lambda^+} w_0(p) = \lambda$, $\lim_{p\to\lambda^-} dw_0(p)/dp = \lim_{p\to\lambda^+} dw_0(p)/dp = 1$, $w_0(p)$ is continuous and differentiable everywhere for $p \in \mathbb{R}_+$.

Anticipating (calculating) the agent's optimal response $\mu^*(w, p)$ the principal chooses w and p that maximize her expected profit rate by solving the optimization problem:

$$\max_{w>0,p>0} \Pi_P(w, p; \mu^*(w, p)) = \max_{w>0,p>0} \left\{ \frac{r\mu^*(w, p)}{\lambda + \mu^*(w, p)} - w + \frac{p\lambda}{\lambda + \mu^*(w, p)} \right\} \tag{3.8}$$

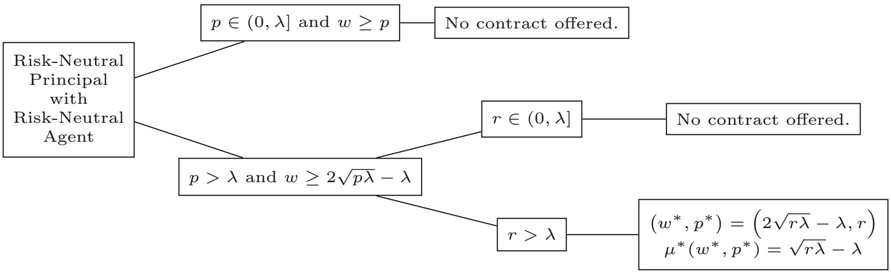

Fig. 3.4 Structure of the proof for Theorem 3.4

with the optimal rates $(w^*, p^*) = \text{argmax}_{w>0,p>0}\, \Pi_P(w, p; \mu^*(w, p))$. We only consider pairs $(w, p) \in \mathbb{R}^2_+$ such that $u_A(\mu; w, p) \geq 0$.

Define: $$\mathfrak{D}_{RN} \equiv \{(w,p) : p \in (0,\lambda], w \geq p\} \cup \left\{(w,p) : p > \lambda, w \geq 2\sqrt{p\lambda} - \lambda\right\} \tag{3.9}$$

Theorem 3.4. *Given a risk-neutral agent as in* (3.2) *and a principal as in* (3.3) *and suppose that* $(w, p) \in \mathfrak{D}_{RN}$.

(a) *If* $r \in (0, \lambda]$, *then the principal does not propose a contract.*
(b) *If* $r > \lambda$, *then the principal's offer and the agent's capacity are respectively*

$$\left(w^*, p^*\right) = \left(2\sqrt{r\lambda} - \lambda, r\right) \text{ and } \mu^*(w^*, p^*) = \sqrt{r\lambda} - \lambda \tag{3.10}$$

resulting in principal's expected profit rate $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{r\lambda} + \lambda$.

Proof. The structure of the proof for Theorem 3.4 is depicted in Fig. 3.4.

Case $p \in (0, \lambda]$ and $w \geq p$: According to Proposition 3.3 part (a), the agent would accept the contract without installing any service capacity. Since $\partial \Pi_P / \partial w = -1 < 0$, the principal chooses $w^* = p$ and $\Pi_P(w^*, p; \mu^*(w^*, p)) = -p + p = 0$. Left with zero expected profit rate, the principal does not propose a contract.

Case $p > \lambda$ and $w \geq 2\sqrt{p\lambda} - \lambda$: According to Proposition 3.3 part (b), the agent accepts the contract and installs capacity $\sqrt{p\lambda} - \lambda$. Since $\partial \Pi_P / \partial w = -1 < 0$, therefore $w^* = 2\sqrt{p\lambda} - \lambda$ and the principal's optimization problem becomes $\max_{p>\lambda} \Pi_P(w^*, p; \mu^*(w^*, p))$ where:

$$\Pi_P(w^*, p; \mu^*(w^*, p)) = r + \lambda - \sqrt{\lambda}\left(\sqrt{p} + \frac{r}{\sqrt{p}}\right) \tag{3.11}$$

Define $x \equiv \sqrt{p}, a \equiv \sqrt{\lambda}$. The principal's expected profit rate, denoted by $f(x)$, can be restated as $f(x) = r + a^2 - a(x + r/x)$ for $x > 0$ and $a > 0$. Maximizing $f(x)$ with respect to $x > 0$ is equivalent to maximizing $\Pi_P(w^*, p; \mu^*(w^*, p))$ with respect to $p > 0$ in the sense that

$$\operatorname*{argmax}_{p>0} \Pi_P(w^*, p; \mu^*(w^*, p)) = \left(\operatorname*{argmax}_{x>0} f(x)\right)^2$$

Denote $p^\star \equiv \text{argmax}_{p>0} \Pi_P(w^*, p; \mu^*(w^*, p))$. Since $d^2f(x)/dx^2 = -2ar/x^3 < 0$, therefore $f(x)$ is concave with respect to $x > 0$ and from the first order condition $df(x)/dx|_{x=x^*} = ar/(x^*)^2 - a = 0 \Rightarrow x^* = \sqrt{r}$. Therefore $p^\star = (x^*)^2 = r$. However $p^\star = r$ is not necessarily the optimal solution because the principal maximizes p for $p > \lambda$. Thus $p^* = \max\{r, \lambda\}$.

Subcase $r \in (0, \lambda]$: $p^* = \lambda$; the principal does not propose a contract since her expected profit rate is zero.

Subcase $r > \lambda$: $p^* = r$; the principal receives $\Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{r\lambda} + \lambda = \left(\sqrt{r} - \sqrt{\lambda}\right)^2 > 0$ and proposes a contract $(w^*, p^*) = (2\sqrt{r\lambda} - \lambda, r)$ that induces the agent to install service capacity $\mu^*(w^*, p^*) = \sqrt{r\lambda} - \lambda$.

In summary, if $r \in (0, \lambda]$, then the principal does not propose a contract (Theorem 3.4 (a)). If $r > \lambda$, then the principal offers $(w^*, p^*) = \left(2\sqrt{r\lambda} - \lambda, r\right)$ and the agent installs capacity $\mu^*(w^*, p^*) = \sqrt{r\lambda} - \lambda$ (Theorem 3.4 (b)), which is an admissible solution according to Definition 2.3. □

Note that in an optimal contract configuration the agent compensates fully the principal for lost revenue during the unit's fail duration.

3.1.1 Sensitivity Analysis of the Optimal Strategy

The principal-agent rationality assumption are odds with the agent accepting a contract offer and responding with $\mu^* = 0$. Therefore the only viable case is when the agent accepts the contract and installs $\mu^*(w, p) = \sqrt{p\lambda} - \lambda$. In this case the rate w is bounded below by $2\sqrt{p\lambda} - \lambda = pP(1) + \mu^*(w, p)$, with $pP(1)$ representing the expected penalty rate charged by the principal when the optimal capacity is installed. It implies that the agent should at least be reimbursed for the expected penalty rate and the cost of the optimal service capacity in exchange for his repair service.

The optimal service capacity itself depends only on p and λ. Note that $\partial\mu^*/\partial p = \sqrt{\lambda/4p} > 0$ and $\partial\mu^*/\partial\lambda = \sqrt{p/4\lambda} - 1$. It indicates that given a λ the agent will increase the μ when the p increases. However, given a p the change in μ^* with respect to the failure rate is not monotonic. The $\sqrt{p\lambda} - \lambda$, as a function of λ, increases when $\lambda \in (0, p/4)$ and decreases when $\lambda \in (p/4, p)$. If the principal's unit is reliable ($\lambda \in (0, p/4)$), then the agent increases the μ when λ increases. If the principal's unit is less reliable ($\lambda \in (p/4, p)$), then the savings from reducing the μ are greater than the increase in p, therefore the agent will reduce μ^* when the λ increases.

The agent's optimal expected utility rate when installing capacity $\mu^*(w,p) = \sqrt{p\lambda} - \lambda$ is $u_A^* \equiv u_A(\mu^*(w,p); w,p) = w - 2\sqrt{p\lambda} + \lambda$, and it depends on w, p and λ. Note that $\partial u_A^*/\partial w = -1 < 0$, $\partial u_A^*/\partial p = -\sqrt{\lambda/p} < 0$, indicating that the agent's optimal expected utility rate decreases with the compensation rate and the penalty rate. Note that $\partial u_A^*/\partial \lambda = -\sqrt{p/\lambda} + 1$, and from Proposition 3.3 $p > \lambda \Rightarrow -\sqrt{p/\lambda} + 1 < 0$, therefore the agent's optimal expected utility rate also decreases with the failure rate.

According to Theorem 3.4, a principal offers a contract to a risk-neutral agent only if $r > \lambda$ and her offer is $(w^*, p^*) = \left(2\sqrt{r\lambda} - \lambda, r\right)$ with expected profit rate $\Pi_P^* \equiv \Pi_P(w^*, p^*; \mu^*(w^*, p^*)) = r - 2\sqrt{r\lambda} + \lambda = \left(\sqrt{r} - \sqrt{\lambda}\right)^2$. The compensation rate and the expected profit rate depend on r and λ, and the penalty rate equals r. Note that $\partial w^*/\partial r = \sqrt{\lambda/r} > 0$ and $\partial w^*/\partial \lambda = \sqrt{r/\lambda} - 1 > 0$ implying that given the λ, the principal will increase w when the revenue rate increases, and given the revenue rate, the principal will increase w when λ increases. Note that $\partial \Pi_P^*/\partial r = \left(\sqrt{r} - \sqrt{\lambda}\right)/\sqrt{r} > 0$ and $\partial \Pi_P^*/\partial \lambda = -\left(\sqrt{r} - \sqrt{\lambda}\right)/\sqrt{\lambda} < 0$. These results imply that given λ, principal's expected profit rate will increase when the revenue rate increases, and given the revenue rate, principal's expected profit rate will decrease when her equipment unit becomes less reliable.

3.1.2 The Second-Best Solution

According to Theorem 3.4, $((w^*, p^*) = \left(2\sqrt{r\lambda} - \lambda, r\right), \mu^*(w^*, p^*) = \sqrt{r\lambda} - \lambda)$ is the second-best solution. When the principal can contract directly on μ there is no moral hazard. Therefore in first-best setting, the agent's expected utility rate, denoted by $u_A^{FB}(w, \mu)$, is simply $u_A^{FB}(w, \mu) = (w - \mu)_+$ for $w > 0$ and $\mu > 0$. Since the principal determines w and μ, her optimization problem is:

$$\max_{w>0,\mu>0} \Pi_P^{FB}(w, \mu) = \max_{w>0,\mu>0} \{rP(0) - w\} = \max_{w>0,\mu>0} \left\{ \frac{r\mu}{\lambda + \mu} - w \right\} \tag{3.12}$$

Denote w^{FB} and μ^{FB} the corresponding solution. Since $\partial \Pi_P^{FB}/\partial w = -1 < 0$, therefore the principal chooses $w^{FB} = \mu$ to ensure the agent's participation and her optimization problem becomes:

$$\max_{\mu>0} \Pi_P^{FB}(\mu) = \max_{\mu>0} \left\{ \frac{r\mu}{\lambda + \mu} - \mu \right\} \tag{3.13}$$

Since $d^2\Pi_P^{FB}(\mu)/d\mu^2 = -2r\lambda/(\lambda+\mu)^3 < 0$, the principal's expected profit rate is concave with respect to $\mu > 0$ and μ^{FB} can be derived from the first order condition $d\Pi_P^{FB}(\mu)/d\mu\big|_{\mu=\mu^{FB}} = r\lambda/\left(\lambda+\mu^{FB}\right)^2 - 1 = 0 \Rightarrow \mu^{FB} = \sqrt{r\lambda} - \lambda$. However $\mu^{FB} = \sqrt{r\lambda} - \lambda$ may not necessarily be the optimal solution because the principal requires $\mu > 0$. Note that $\mu^{FB} = \sqrt{\lambda}\left(\sqrt{r} - \sqrt{\lambda}\right) > 0$ only if $r > \lambda$. Therefore the first-best solution is:

$$w^{FB} = \mu^{FB} = \sqrt{r\lambda} - \lambda \text{ for } r > \lambda \tag{3.14}$$

By comparing the second-best solution (3.10) to the first-best solution (3.14), we conclude:

1. The principal offers a contract only when $r > \lambda$ indicating that the existence of a beneficial contract for risk-neutral agent is determined exogenously by the market (the revenue rate r) and the nature of the equipment (the failure rate λ), which is consistent with Proposition 2 in Harris and Raviv (1978).
2. The proposed w in the second-best solution is higher than that in the first-best solution ($w^* = 2\sqrt{r\lambda} - \lambda > \sqrt{r\lambda} - \lambda = w^{FB}$), because the principal has to compensate for the p when the agent's μ is not observable. Nevertheless, the second-best contract is efficient (as the first-best contract) because of point 3 below.
3. The optimal capacity in the first-best solution and the second-best solution are the same ($\mu^{FB} = \mu^*(w^*, p^*) = \sqrt{r\lambda} - \lambda$), indicating that the principal can induce a risk-neutral agent to install the desired capacity without contracting on it directly. Furthermore, the principal receives the same expected profit rate no matter if the agent's action is observable (thus contractible) or not. This is consistent with Proposition 3 part (i) in Harris and Raviv (1978).
4. Finally when the agent is risk-neutral, the principal is guaranteed getting the revenue rate r at all times regardless of the state of the equipment unit (because $p^* = r$). This comes at the cost of the contract ($w^* = 2\sqrt{r\lambda} - \lambda$). In other words, the principal's profit rate appears as if it is deterministic. However this is not true for a risk-averse agent, as seen in Chap. 4.

3.1.3 Our Principal-Agent Game

To clarify the interplay of decisions by the principal and the agent, we cast the principal-agent problem in an extensive form game depicted in Fig. 3.5 below, where "P" represents the principal and "A" the agent.

There are four possible strategies the principal can choose from:

O_1: Offer a contract with $p \in (0, \lambda]$ and $w \in (0, p)$.
O_2: Offer a contract with $p \in (0, \lambda]$ and $w \geq p$.

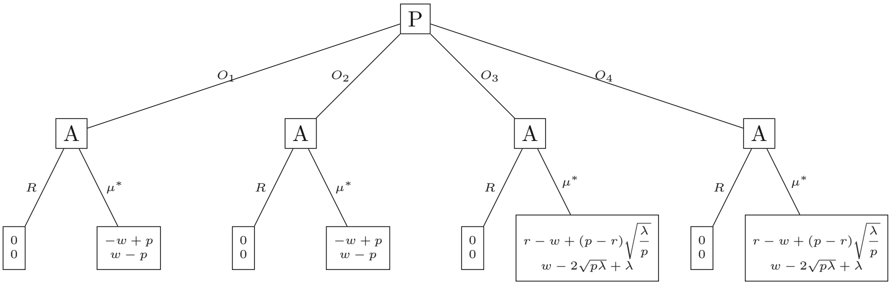

Fig. 3.5 Structure of the principal-agent extensive form game

O_3: Offer a contract with $p > \lambda$ and $w \in \left(0, 2\sqrt{p\lambda} - \lambda\right)$.

O_4: Offer a contract with $p > \lambda$ and $w \geq 2\sqrt{p\lambda} - \lambda$.

For any contract offer by the principal, there are two strategies for the agent to choose from: "R" when rejecting the contract, and μ^* for accepting the contract and installing the service capacity that maximizes the agent's expected profit rate. If the principal offers O_1 or O_2 and the agent accepts the contract, then $\mu^* = 0$. If the principal offers O_3 and O_4 and the agent accepts the contract, then $\mu^* = \sqrt{p\lambda} - \lambda$.

The principal's expected profit rate and the agent's expected utility rate are presented in the leaves of the tree in Fig. 3.5. The element above and below are the principal's and the agent's values respectively.

The agent would accept the contract only if his maximized expected utility rate is no less than his reservation utility rate $\underline{u_A} = 0$, therefore the agent accepts the contract when the principal offers O_2 and O_4, and rejects the contract when the principal offers O_1 and O_3. The principal always prefers the agent to accept the contract and install a positive service capacity. Therefore the principal would choose O_4 to all other options. Thus there is only one (subgame perfect) Nash equilibrium: the principal offers a contract with $p > \lambda$ and $w \geq 2\sqrt{p\lambda} - \lambda$ and the agent accepts the contract and installs $\mu^* = \sqrt{p\lambda} - \lambda > 0$.